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Mode III near and far fields for a crack lying in or along a joint

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Abstract

Matched asymptotic expansions allow one to give, at two scales, a description of the solution to the antiplane problem of two different elastic substrates bonded together by a thin elastic adhesive layer with a crack lying inside or even outside at a short distance of it. The leading term is the classical solution obtained, by finite elements for instance, when perfect transmission conditions are assumed through the line modelling the interface. The corrections depend on the relative stiffnesses of the components and on the thickness of the joint. It also depends on the crack location, within the joint or outside. Two kinds of Mode III stress intensity factors are defined, a fictitious and an actual one, they are so-called far and near (or remote and local) by the authors. The fictitious one is meaningless and must be rescaled by an appropriate coefficient to give the actual one.

1. Introduction

Structures are often made of complex assemblies, among other methods, gluing and welding are used in some cases to realize the bondings. They imply the existence of a thin additional layer made of an extraneous material. In general, it is difficult to take into account these small zones, in a structural computation by finite elements for instance, they are treated as idealized surfaces with perfect transmission of displacements and forces between the components. As will be seen below, this approximation is in general, quite realistic, it corresponds to the leading term of an asymptotic expansion of the solution in terms of the dimensionless parameter associated with the joint thickness. Nevertheless, such a description becomes insufficient if fracture or any micro-mechanism of the bonding zone is involved. A more precise knowledge of the stress field within and near the joint is required.

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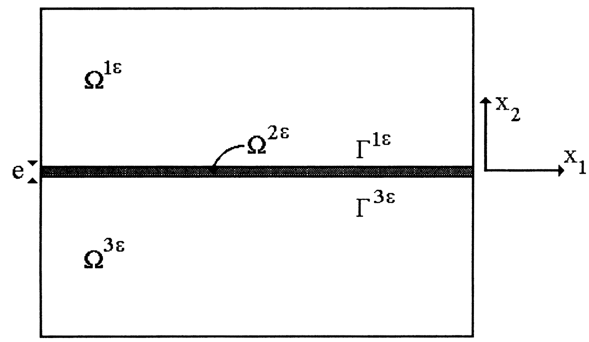

Fig. 1. The two substrates $\Omega^{1\varepsilon}$ and $\Omega^{3\varepsilon}$ bonded together by an adhesive layer $\Omega^{2\varepsilon}$.

A first approach initiated by the problems of gluing is based on an analysis of the stress field within the joint (Goland and Reissner, 1944; Wooley and Carver, 1971; Gilibert and Rigolot, 1979; Delale et al., 1981; Tsai and Morton, 1994). The assumption of slender substrates is often made and is then dedicated to plate assembly.

Another approach consists in replacing the joint by springs and deriving a rheological model (Cornell, 1953; Jones and Whittier, 1967; Léné and Leguillon, 1982; Rose, 1987; Geymonat and Krasucki, 1996). This model embeds a major assumption, the joint stiffness is much smaller than the substrate ones.

Other points of view are also proposed. Hutchinson et al. (1987) considered the plane elasticity problem of two substrates bonded together along a line with a crack paralleling the interface. The analysis on an infinite domain with remote loads exhibits a linear relation between far and near stress intensity factors (as baptised by Ryvkin, see hereafter). A similar linear relation is obtained by Suo and Hutchinson (1989) for a crack lying along the surface of a thin layer in a sandwich structure. Ryvkin et al. (1995) studies the interaction of a thin layer and a crack parallel to it, far and near Mode III stress intensity factors are defined, they are interpreted in terms of an energy release rate, it leads to the conclusion that the far one may become meaningless. Ryvkin (1996) assumes a periodic stacking sequence in a laminated composite and examines the dependence of the Mode III stress intensity factor on the layers thickness. Atkinson and Chen (1996) consider a structure made of two identical elastic or viscoelastic substrates bonded together by an elastic or viscoelastic adhesive. They show the influence of the thickness and the stiffness of the adhesive on the Mode III stress intensity factor.

The present study, using matched asymptotics (Nguetseng and Sanchez-Palencia, 1985), allows one to give a two scales analysis of the problem of two different elastic substrates bonded together by a thin elastic layer (smallness of this layer is the only assumption) when a crack lies within the joint (cohesive fracture) or along the interfaces between the joint and the substrates (adhesive fracture) or even outside the joint at a short distance. The leading term is the classical one obtained for instance by a finite element calculation with perfect transmission conditions through the line modelling the adhesive and stress free conditions on the crack lips. The next terms of the expansions arise to be corrections to this coarse approximation. The role of the components stiffness and the thickness of the adhesive are explicitly expressed. Two kinds of stress intensity factors can be derived from this analysis but rather than calling them far and near stress factors (Ryvkin et al., 1995), we prefer talking about fictitious and actual. For the sake of simplicity, the analysis is restricted to an out of plane bidimensional elastic problem without body forces, but the same reasoning extends to the generalized plane elasticity without real insuperable difficulties (a brief generalization is proposed in Leguillon, 1995). The complete structure Ω^{ε} is divided into two parts $\Omega^{1\varepsilon}$ and $\Omega^{3\varepsilon}$ made of two different isotropic materials (shear modulus G_1 and G_3 also noted sometimes as G_+ and G_-) bonded together along the straight boundaries

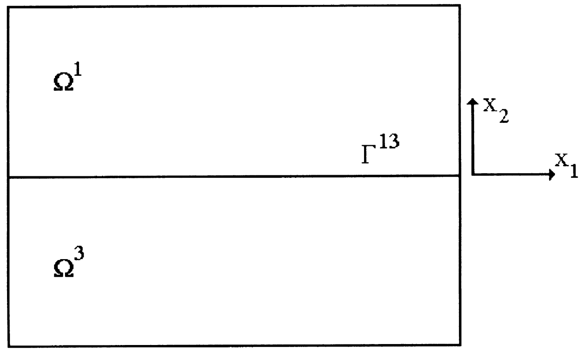

Fig. 2. The limit unperturbed domain $\Omega = \Omega^1 \cup \Omega^3$.

$\Gamma^{1\varepsilon}$ and $\Gamma^{3\varepsilon}$ parallel to the x_1-axis by a thin layer $\Omega^{2\varepsilon}$ with thickness e (Fig. 1). The stiffness G_2 of the adhesive can be larger (oxide in metal matrix materials) or smaller (polymer in composite bondings).

The dimensionless small parameter ε is the ratio e/L where L denotes a characteristic length of the structure, the total joint length for instance.

Remark 1. In the following $L = 1$ for simplicity, it can be understood as a way to define dimensionless lengths, in other words, each physical length and displacement is divided by L.

Throughout this paper the words ‘joint’, ‘adhesive layer’, and ‘bonding zone’ will denote the physical layer with a small but non zero thickness, it can also be understood as an ‘interphase’, whereas the word ‘interface’ will be used for idealized contact surfaces or lines in 2-D (without thickness) between different materials. Moreover, for a better understanding, it is recalled that the adjectives outer and inner (terms, expansions, ...) are used to characterize, respectively, the far and near fields.

2. The interface boundary layer

2.1. Outer expansion

As $\varepsilon \to 0$, the domain Ω^ε reduces to a domain Ω made of two parts Ω^1 and Ω^3 sharing the common boundary Γ^{13} (Fig. 2).

The adhesive material has disappeared in this final frame which looks typically like the one used by an engineer starting to mesh the structure before a finite elements computation. Nevertheless, the transmission conditions to write down along this fictitious common interface are not obvious. Let us assume an outer expansion of the out of plane displacement solution U^ε in the form

$$U^\varepsilon(x_1, x_2) = U^0(x_1, x_2) + f_1(\varepsilon) U^1(x_1, x_2) + \cdots, \tag{1}$$

where $f_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ and where U^0 is the solution to the unperturbed problem settled on Ω with the equilibrium equation and classical perfect transmission conditions through Γ^{13}

$$\frac{\partial^2 U^0}{\partial x_1^2} + \frac{\partial^2 U^0}{\partial x_2^2} = 0 \quad \text{in } \Omega^1 \quad \text{and} \quad \Omega^3, \tag{2}$$

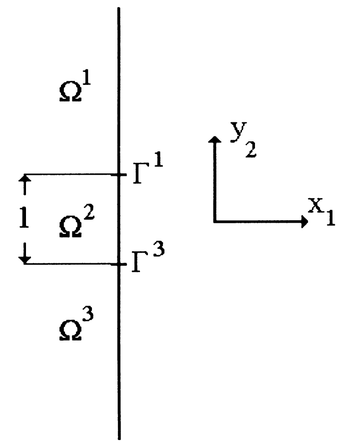

Fig. 3. The inner domain derived from stretching $y_2 = x_2/\varepsilon$.

$$[\![U^0]\!] = 0, \quad [\![\sigma^0]\!]\underline{n} = 0 \quad \text{through } \Gamma^{13}, \tag{3}$$

where $[\![\cdot]\!]$ denotes the jump of any quantity through Γ^{13} and $\underline{n}$ is the normal to Γ^{13} (its direction is chosen arbitrarily and determines the sign of the jumps). The stress field σ^0 is defined in the two substrates Ω^k ($k = 1, 3$) by

$$\sigma^0_{i3} = G_k \frac{\partial U^0}{\partial x_i}. \tag{4}$$

Remark 2. No boundary conditions are mentioned herein since they play no specific role. There is just one simplifying assumption to consider, prescribed forces or displacements are acting only at a distance of the ends of the bonding zone.

Remark 3. Transmission conditions are a priori prescribed in (3). It will be seen below that otherwise they will occur as a consequence of the matching process.

U^0 and U^1 are defined on the limit domain Ω and thus provide the explanation for the name 'outer', the expansion (1) is only valid out of a vicinity of the interface Γ^{13}.

Obviously U^1 cannot be straightforwardly determined. The matching conditions will govern the behaviour of the first additional correction to the unperturbed term. To this aim, let us consider an interior point P of the interface Γ^{13} (i.e. any point except the ends). U^0 is smooth in the vicinity points, moreover, at this time, we assume that the other terms of (1) are also piecewise smooth, i.e. smooth above and under the interface (it will be a posteriori checked). Thus, they can be expanded in Taylor series with respect to the space variable x_2 orthogonal to Γ^{13}

$$U^j_\pm(x_1, x_2) = U^j_\pm(x_1, 0) + x_2 \frac{\partial U^j_\pm}{\partial x_2}(x_1, 0) + \frac{x_2^2}{2}\frac{\partial^2 U^j_\pm}{\partial x_2^2}(x_1, 0) + \cdots, \tag{5}$$

where the index $\pm$ stands, respectively, for the upper ($x_2 > 0$) and lower ($x_2 < 0$) domains. In particular, due to the definition of U^0

$$U^0_+(x_1, 0) = U^0_-(x_1, 0). \tag{6}$$

Expansion (5) determines the inner behaviour of the outer terms.

2.2. Inner expansion, terms 0 and 1

Next, in order to define the inner expansion, let us consider in the vicinity of P the stretched variable $y_2 = x_2/\varepsilon$. As $\varepsilon \to 0$, the domain spanned by y_2 is an infinite line, the upper part lies in Ω^1, the lower one in Ω^3 and the intermediate one with a unit length in Ω^2, it is the stretched adhesive layer bounded up and down by the points Γ^1 and Γ^3 (Fig. 3).

The inner expansion reads

$$U^\varepsilon(x_1, \varepsilon y_2) = F_0(\varepsilon)V^0(x_1, y_2) + F_1(\varepsilon)V^1(x_1, y_2) + F_2(\varepsilon)V^2(x_1, y_2) + \cdots. \tag{7}$$

Each term of this expansion must fulfil the equilibrium equation and classical continuity conditions through Γ^1 and Γ^3 located, respectively, at $y_2 = 1/2$ and $y_2 = -1/2$. Moreover, the matching conditions govern the behaviour at infinity. At the leading order the system writes

$$\frac{\partial^2 V^0}{\partial y_2^2} = 0, \tag{8}$$

$$[\![V^0]\!] = 0, \quad [\![\Sigma^0]\!]\underline{n} = 0 \quad \text{for } y_2 = \pm 1/2, \tag{9}$$

$$V^0(x_1, y_2) \sim U^0(x_1, 0) \quad \text{as } y_2 \to \pm\infty, \tag{10}$$

where the stress field Σ^0 is defined by

$$\Sigma^0_{13} = 0, \quad \Sigma^0_{23} = G_k \frac{\partial V^0}{\partial y_2} \quad \text{in } \Omega^k, \quad k = 1, 2, 3.$$

Finally, the first term of (7) is given by

$$F_0(\varepsilon) = 1, \quad V^0(x_1, y_2) = U^0(x_1, 0). \tag{11}$$

Remark 4. If the continuity of the outer term (3) was not assumed it will appear at this step as a consequence of the continuity of V^0. See Remark 3 concerning a priori and a posteriori defined transmission conditions.

At the next order, V^1 arises to be piecewise linear, the equilibrium equation is identical to (8) and the continuity condition is similar to (9)

$$[\![\Sigma^1]\!]n = 0 \quad \text{for } y_2 = \pm 1/2,$$

with

$$\Sigma^1_{13} = G_k \frac{\partial V^0}{\partial x_1}, \quad \Sigma^1_{23} = G_k \frac{\partial V^1}{\partial y_2} \quad \text{in } \Omega^k, \quad k = 1, 2, 3.$$

The matching conditions now read

$$V^1_{\pm}(x_1, y_2) \sim y_2 \frac{\partial U^0_{\pm}(x_1, 0)}{\partial x_2} + U^1_{\pm}(x_1, 0) \quad \text{as } y_2 \rightarrow \pm\infty,$$

and lead to

$$F_1(\varepsilon) = \varepsilon.$$

The corresponding term of the inner asymptotic expansion writes

$$V^1(x_1, y_2) = y_2 \frac{\partial U^0_{\pm}}{\partial x_2}(x_1, 0) + U^1_{\pm}(x_1, 0) \quad \text{resp. in } \Omega^1\ (+) \quad \text{and} \quad \Omega^3\ (-), \tag{12}$$

$$V^1(x_1, y_2) = y_2 \frac{\partial U^0_{+}}{\partial x_2}(x_1, 0) + U^1_{+}(x_1, 0) + \frac{G_1 - G_2}{2G_2} \frac{\partial U^0_{+}}{\partial x_2}(x_1, 0)(2y_2 - 1), \tag{13}$$

$$V^1(x_1, y_2) = y_2 \frac{\partial U^0_{-}}{\partial x_2}(x_1, 0) + U^1_{-}(x_1, 0) + \frac{G_3 - G_2}{2G_2} \frac{\partial U^0_{-}}{\partial x_2}(x_1, 0)(2y_2 + 1), \tag{14}$$

in Ω^2 resp. for $y_2 > 0$ and $y_2 < 0$.

Remark 5. The normal stress continuity is ensured through the line $y_2 = 0$ as a consequence of the normal stress continuity of the outer term U^0 (3). In other words, if such a continuity (3) was not assumed it will arise here as a consequence. See again Remarks 3 and 4 concerning a priori and a posteriori defined transmission conditions.

On the other hand, the displacements continuity must be added as a condition

$$[\![U^1]\!] = \frac{G_1 - G_2}{2G_2} \frac{\partial U^0_{+}}{\partial x_2}(x_1, 0) + \frac{G_3 - G_2}{2G_2} \frac{\partial U^0_{-}}{\partial x_2}(x_1, 0). \tag{15}$$

As previously defined, the brackets denote a jump, $[\![U^1]\!] = U^1_{+}(x_1, 0) - U^1_{-}(x_1, 0)$, thus, the above condition (15) is the first transmission condition through Γ^{13} for the second term U^1 of the outer expansion (1).

2.3. Inner expansion, term 2

One additional condition through Γ^{13} is missing in order to have a well-posed problem defining U^1. It will be obtained from investigating the next term V^2 of the inner expansion (7). The equilibrium equation reads

$$\frac{\partial^2 V^2}{\partial y_2^2} + \frac{\partial^2 V^0}{\partial x_1^2} = 0 \quad \text{in } \Omega^k, \quad k = 1, 2, 3,$$

and becomes, with the help of the macroscopic equilibrium equation (2) and the knowledge of the first

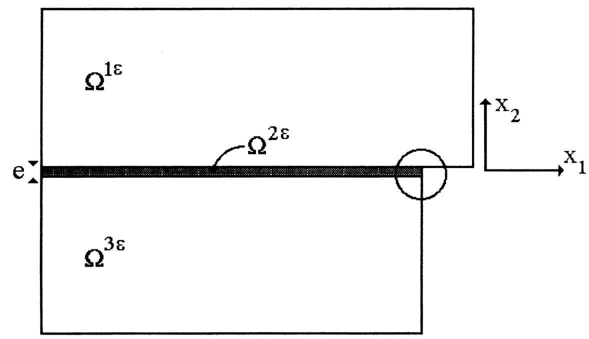

Fig. 4. A singular corner at the end of the adhesive layer.

inner term (11)

$$\frac{\partial^2 V^2}{\partial y_2^2}(x_1, y_2) = \frac{\partial^2 U^0}{\partial x_2^2}(x_1, 0). \tag{16}$$

Moreover, again as a consequence of the equilibrium equation (2) and the continuity of U^0 (3) and (6), upper and lower values of the right-hand-side member of (16) are equal, the index $\pm$ is not required. The transmission conditions (9) remain unchanged (up to the index) and the matching conditions impose the behaviour at infinity

$$V^2(x_1, y_2) \sim \frac{y_2^2}{2}\frac{\partial^2 U^0}{\partial x_2^2}(x_1, 0) + y_2\frac{\partial U^1_\pm}{\partial x_2}(x_1, 0) + U^2_\pm(x_1, 0) \quad \text{as } y_2 \to \pm\infty.$$

The matching conditions imply

$$F_2(\varepsilon) = \varepsilon^2,$$

and V^2 is a quadratic function defined by

$$V^2(x_1, y_2) = \frac{y_2^2}{2}\frac{\partial^2 U^0}{\partial x_2^2}(x_1, 0) + y_2\frac{\partial U^1_\pm}{\partial x_2}(x_1, 0) + U^2_\pm(x_1, 0), \tag{17}$$

in Ω^1 (+) resp. Ω^3 (−).

$$V^2(x_1, y_2) = [id] + \left(\frac{G_1 - G_2}{4G_2}\frac{\partial^2 U^0}{\partial x_2^2} + \frac{G_1 - G_2}{2G_2}\frac{\partial U^1_+}{\partial x_2}\right)(x_1, 0)(2y_2 - 1), \tag{18}$$

$$V^2(x_1, y_2) = [id] + \left(\frac{G_2 - G_3}{4G_2}\frac{\partial^2 U^0}{\partial x_2^2} + \frac{G_3 - G_2}{2G_2}\frac{\partial U^1_-}{\partial x_2}\right)(x_1, 0)(2y_2 + 1), \tag{19}$$

in Ω^2 resp. for $y_2 > 0$ and $y_2 < 0$.

In the above two relations, the bracket $[id]$ must be replaced by an expression identical to (17) [this feature was already met in (12)–(14)].

There are now two continuity conditions through the line $y_2 = 0$. The first one provides a relation on

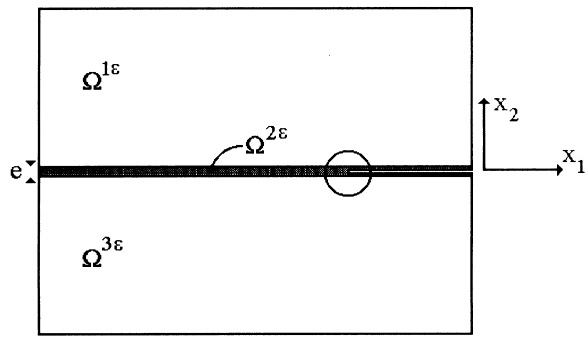

Fig. 5. A singular crack tip for a crack lying within the adhesive layer.

the discontinuity of U^2 through Γ^{13} which becomes relevant when studying this term (the third outer one), and the second one provides as expected the complementary condition to (15) relative to U^1

$$[\![\sigma^1]\!]\underline{n} = \left(G_2 - \frac{G_1 + G_3}{2}\right)\frac{\partial^2 U^0}{\partial x_2^2}(x_1, 0). \tag{20}$$

The stress field σ^1 is defined by a relation analogous to (4) (up to the index once more) and

$$[\![\sigma^1]\!]\underline{n} = G_1\frac{\partial U_+^1}{\partial x_2}(x_1, 0) - G_3\frac{\partial U_-^1}{\partial x_2}(x_1, 0).$$

The two conditions (15) and (20) are consistent but requires U^0 to be smooth enough since, roughly speaking, U^1 depends on the first derivative of U^0 and the first derivative of U^1 depends on the second derivative of U^0. Such a smoothness holds true at any interior point of the straight interface but generally fails at the ends of this line, at a sharp corner or at a crack tip (Figs. 4 and 5).

The forthcoming sections of this paper deal with the analysis of this second situation (Fig. 5) which is the most entangled (Leguillon, 1994, 1995, 1996).

2.4. The low stiffness joint

If one assumes the stiffness of the bonding material to be far smaller than that of the substrates, the following relation

$$G_2 = \varepsilon G, \tag{21}$$

where G takes a finite value close to G_1 and G_3, is commonly admitted.

Then the above asymptotics are obviously modified. Eqs. (8) and (10) remain unchanged as well as the first part of (9). But the stress continuity condition at $y_2 = \pm 1/2$ combined with the asymptotics and (21) entail a relation between two successive expansion orders $j-1$ and j (in this subsection, a single $\tilde{}$ is used to denote the terms of the former model and a double $\tilde{\tilde{}}$ those of the springs model)

$$G_1\frac{\partial \tilde{\tilde{V}}_+^0}{\partial y_2} = G_3\frac{\partial \tilde{\tilde{V}}_-^0}{\partial y_2} = 0, \quad G_1\frac{\partial \tilde{\tilde{V}}_+^j}{\partial y_2} = G\frac{\partial \tilde{\tilde{V}}_*^{j-1}}{\partial y_2}, \quad G_3\frac{\partial \tilde{\tilde{V}}_-^j}{\partial y_2} = G\frac{\partial \tilde{\tilde{V}}_*^{j-1}}{\partial y_2} \quad \text{for } j \geq 1.$$

$\tilde{\tilde{V}}_\pm^j$ and $\tilde{\tilde{V}}_*^j$ denote $\tilde{\tilde{V}}^j$ in Ω_1 (+), resp. Ω_3 (−) and Ω_2 (*). Remark 4 becomes obsolete, the continuity condition on $\tilde{\tilde{U}}^0$ is no longer a necessary condition. One has to consider the upper and lower values $\tilde{\tilde{U}}^0_{\pm}$ and the first term of the inner expansion reads

$$\tilde{\tilde{V}}^0_{\pm}(x_1, y_2) = \tilde{\tilde{U}}^0_{\pm}(x_1, 0), \quad \tilde{\tilde{V}}^0_{*}(x_1, y_2) = [\![\tilde{\tilde{U}}^0]\!] y_2 + \langle \tilde{\tilde{U}}^0 \rangle$$

instead of (11), where

$$[\![\tilde{\tilde{U}}^0]\!] = \tilde{\tilde{U}}^0_{+}(x_1, 0) - \tilde{\tilde{U}}^0_{-}(x_1, 0), \quad \langle \tilde{\tilde{U}}^0 \rangle = \frac{\tilde{\tilde{U}}^0_{+}(x_1, 0) + \tilde{\tilde{U}}^0_{-}(x_1, 0)}{2}.$$

The stress continuity condition at the next order $\tilde{\tilde{V}}^1$ leads to

$$G_1 \frac{\partial \tilde{\tilde{U}}^0_{+}}{\partial x_2}(x_1, 0) = G_3 \frac{\partial \tilde{\tilde{U}}^0_{-}}{\partial x_2}(x_1, 0) = \tilde{\tilde{\sigma}}^0 \underline{n} = G [\![\tilde{\tilde{U}}^0]\!].$$

This is the well-known springs model involving a relation between the jump of the leading term $\tilde{\tilde{U}}^0$ through Γ^{13} and the (continuous) normal stress $\tilde{\tilde{\sigma}}^0 \underline{n}$. Coefficient $G = G_2/e$ [i.e. relation (21) in the present dimensionless model, see Remark 1] is the spring stiffness.

It differs from the model settled before, and it is obviously difficult to introduce the asymptotic relation (21) in the former model to recover the latter one. Two small parameters are competing, the relative joint thickness e/L and the relative joint stiffness G_2/G_1 or G_2/G_3. However, it can be very formally done. $\tilde{U}^0$ remains continuous as stated in the assumptions, and the following jump relation

$$\tilde{\sigma}^0 \underline{n} = G [\![\tilde{U}^0 + \varepsilon \tilde{U}^1]\!]$$

is derived from (15). Investigating the next terms of the asymptotics leads to extend this relation to

$$(\tilde{\sigma}^0 + \varepsilon \tilde{\sigma}^1)\underline{n} = G [\![\tilde{U}^0 + \varepsilon \tilde{U}^1 + \varepsilon^2 \tilde{U}^2]\!], \ \ldots$$

which incites one to consider the latter model as a limit of the former one.

To illustrate that point let us study the following simple out of plane shear problem. The two substrates are identical $G_1 = G_3 = G$ and $G_2 = \varepsilon G$ [i.e. (21) holds true]. The domain Ω^{ε} (Fig. 1) is submitted to an out of plane traction T on the upper face and $-T$ on the lower one, thus $x_2 = 0$ is a symmetry axis and we consider only the solution in the upper part $x_2 > 0$ of the domains under consideration. The exact solution (in the perturbed domain) is

$$U^{\varepsilon} = \frac{T}{\varepsilon G} x_2 \quad \text{for } 0 < x_2 < \varepsilon/2, \quad U^{\varepsilon} = \frac{T}{G}\left(x_2 + \frac{1-\varepsilon}{2}\right) \quad \text{for } \varepsilon/2 < x_2,$$

and the approximations corresponding to the two models (in the unperturbed domain) read

$$\tilde{U}^0 = \frac{T}{G} x_2, \quad \tilde{U}^1 = \frac{1-\varepsilon}{2\varepsilon} \frac{T}{G}, \quad \tilde{\tilde{U}}^0 = \frac{T}{G}(x_2 + 1/2).$$

In this elementary example, the springs model leading term $\tilde{\tilde{U}}^0$ appears clearly as the limit of the two first terms $\tilde{U}^0 + \varepsilon \tilde{U}^1$ of the former model as $\varepsilon \to 0$. Moreover, the former model is confounded with the exact solution whatever ε.

A numerical comparison between the different models is proposed in Section 6 on a structure embedding a crack.

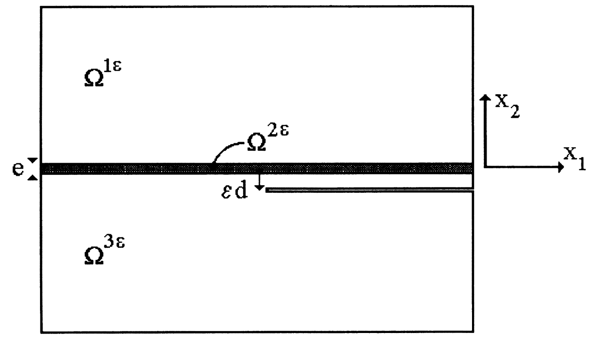

Fig. 6. A crack paralleling the adhesive layer at a distance εd.

Remark 6. Counter to the springs model, the present model works for any value of the joint stiffness, in the above elementary example the jump is positive if the joint is more compliant than the substrates triggering a gap, and it is negative if the joint is stiffer leading to an overlap.

One difference between the two models, which plays an important role in the next section, lies in the stress concentration due to a crack tip for instance. In the former case the crack tip singularity exists in the leading term of the outer expansion, whereas in the springs model, the singularity is weak (logarithmic) and concerns only the stress component $\underset{\sim}{\sigma}{}^{0}_{11}$ parallel to the crack. In the last case, the brittle fracture criteria do not apply to the outer term $\tilde{U}^0$ while they remain valid in the other one.

3. The crack lips—outer and inner expansions

It is now assumed that a crack extends within the joint. It can be located whether inside the bonding material (cohesive fracture) or along the upper or lower contact zones between the adhesive layer and the substrates (adhesive fracture), or even at a short distance within one of the substrate (Fig. 6).

As will be seen in the next section, the exact location of this crack does not play a major role at the beginning of the analysis. It will be assumed here that the crack lies at a distance εd from the midpoint of the joint (i.e. the distance smallness is comparable to the joint thickness).

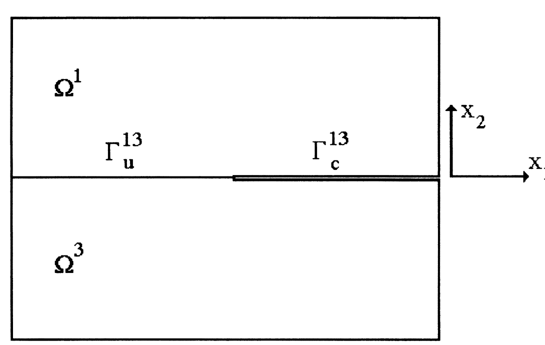

Fig. 7. The crack in the limit domain $\Omega = \Omega^1 \cup \Omega^3$.

As before, as $\varepsilon \rightarrow 0$ the domain Ω^{ε} reduces to Ω which now embeds a crack along the interface Γ^{13}. This interface has two distinct complementary parts, the crack Γ_{c}^{13} and Γ_{u}^{13} corresponding to the uncracked part (which was the aim of the two preceding sections). It is to be pointed out that εd vanishes as $\varepsilon \rightarrow 0$ and then the exact location of the crack does not play any role in this framework (Fig. 7).

To complete the description of the solution, the same previous reasoning must be done at any interior point P' of the crack lips, whether on the upper or on the lower lip (denoted, respectively, by the index $+$ and $-$), the two situations are symmetric. These lips are assumed to be stress free and conditions (3) must be replaced by

$$\sigma_{\pm}^{0} \underline{n} = 0 \quad \text{on } \Gamma_{\mathrm{c}\pm}^{13},$$

which rewrites simply

$$G_{\pm} \frac{\partial U_{\pm}^{0}}{\partial x_2}(x_1, 0) = 0 \quad \text{on } \Gamma_{\mathrm{c}\pm}^{13}.$$

Note that since no continuity condition is required through the crack, $U_{+}^{0}(x_1, 0)$ and $U_{-}^{0}(x_1, 0)$ are distinct. The analysis of the inner expansion (7) is now easier, the first term is

$$V^{0}(x_1, y_2) = U_{\pm}^{0}(x_1, 0) \quad \text{resp. for } d \le y_2\ (+) \quad \text{and} \quad y_2 \le d\ (-).$$

It holds true whatever the relative location of the crack, as well as the second term which reads

$$V^{1}(x_1, y_2) = U_{\pm}^{1}(x_1, 0) \quad \text{resp. for } d \le y_2\ (+) \quad \text{and} \quad y_2 \le d\ (-).$$

The third term V^{2} is somewhat similar to the previous one but depends now on the location of the crack.

- If $-1/2 \le d \le 1/2$, i.e. the crack is inside or along one side of the joint, then (17) holds true in Ω^{1} and Ω^{3}. Eq. (18) holds in the upper strip $d \le y_2 \le 1/2$ and (19) in the lower one $-1/2 \le y_2 \le d$. The continuity conditions (20) must be replaced by a stress free condition along the lips defined by $y_2 = d$, it leads to

$$G_1 \frac{\partial U_{+}^{1}}{\partial x_2}(x_1, 0) = \left(\frac{G_2 - G_1}{2} - dG_2 \right) \frac{\partial^2 U_{+}^{0}}{\partial^2 x_2^2}(x_1, 0), \tag{22}$$

$$G_3 \frac{\partial U_{-}^{1}}{\partial x_2}(x_1, 0) = \left(\frac{G_3 - G_2}{2} - dG_2 \right) \frac{\partial^2 U_{-}^{0}}{\partial^2 x_2^2}(x_1, 0), \tag{23}$$

  which are non-homogeneous boundary conditions for the problem U^{1}.
- If $d \ge 1/2$ the crack is outside the joint in the upper domain then, (17) with the index $+$ holds true above the crack and the stress free boundary condition at $y_2 = d$ leads to

$$G_1 \frac{\partial U_{+}^{1}}{\partial x_2}(x_1, 0) = -dG_1 \frac{\partial^2 U_{+}^{0}}{\partial^2 x_2^2}(x_1, 0). \tag{24}$$

  Eq. (17) with the index $-$ also holds true below the crack in Ω^{3}. Eq. (19) holds true in the whole joint from $y_2 = -1/2$ to $y_2 = 1/2$ (the index $-$ must be added to the second-order derivative $\partial^2 U^{0}/\partial x_2^2$). In the layer located between the joint and the crack $1/2 \le y_2 \le d$, the solution takes the general form

$$V^2(x_1, y_2) = \frac{y_2^2}{2}\frac{\partial^2 U_-^0}{\partial x_2^2}(x_1, 0) + by_2 + c.$$

The stress free condition on the lower lip of the crack leads to

$$b = -d\frac{\partial^2 U_-^0}{\partial x_2^2}(x_1, 0).$$

The continuity condition through the interface Γ^1 allows one to determine the constant c but is irrelevant for this part of the analysis and finally it is the normal stress continuity condition through Γ^1 which provides the lower boundary condition for U^1 along the crack

$$G_3\frac{\partial U_-^1}{\partial x_2}(x_1, 0) = \left(\frac{G_1 + G_3 - 2G_2}{2} - dG_1\right)\frac{\partial^2 U_-^0}{\partial x_2^2}(x_1, 0). \tag{25}$$

- Very symmetrically, if the crack is in the lower substrate, $d \leq -1/2$, the same reasoning yields

$$G_1\frac{\partial U_+^1}{\partial x_2}(x_1, 0) = \left(\frac{2G_2 - G_1 - G_3}{2} - dG_3\right)\frac{\partial^2 U_+^0}{\partial x_2^2}(x_1, 0), \tag{26}$$

$$G_3\frac{\partial U_-^1}{\partial x_2}(x_1, 0) = -dG_3\frac{\partial^2 U_-^0}{\partial^2 x_2^2}(x_1, 0). \tag{27}$$

Summing up these results gives the general form of the boundary conditions along the crack lips

$$G_1\frac{\partial U_+^1}{\partial x_2}(x_1, 0) = H_+(d)\frac{\partial^2 U_+^0}{\partial x_2^2}(x_1, 0), \qquad G_3\frac{\partial U_-^1}{\partial x_2}(x_1, 0) = H_-(d)\frac{\partial^2 U_-^0}{\partial x_2^2}(x_1, 0), \tag{28}$$

where $H_+(d)$ and $H_-(d)$ depend on the location of the crack and are given, respectively, by (22), (24), (26) and (23), (25), (27).

4. The crack tip singularity and the second outer term

U^0 is the solution to a problem settled on a bimaterial structure with a crack lying along the interface. The two components being isotropic, the generalized plane elasticity solution $\underline{U}$ (the underbar denotes a vector) undergoes a singular displacement developed in the vicinity of the crack tip under the form (Williams, 1959)

$$\underline{U}(x_1, x_2) = \underline{U}(0, 0) + \Re e(k_c r^{\lambda_c}\underline{u}_c(\theta)) + k_{\mathrm{III}}\sqrt{r}u_{\mathrm{III}}(\theta)\underline{e}_3 + \cdots, \tag{29}$$

where r and θ are the polar coordinates with the origin at the crack tip and $\underline{e}_3$ is the unit normal to the (x_1, x_2) plane. The complex exponent λ_c is such that $\Re e(\lambda_c) = 1/2$, k_c is a complex intensity factor and $\underline{u}_c$ is the complex plane crack tip mode. The next term, dedicated to the out of plane Mode III involves only real terms, the intensity factor k_{III} and the mode

$$U_{\mathrm{III}+}(x_1, x_2) = \sqrt{r}u_{\mathrm{III}}(\theta), \tag{30}$$

with

$$u_{\mathrm{III}}(\theta) = \cos(\theta/2) \quad \text{in } \Omega^1, \quad u_{\mathrm{III}}(\theta) = \frac{G_1}{G_3}\cos(\theta/2) \quad \text{in } \Omega^3. \tag{31}$$

The role of the index + in (30) will appear below, it is not related to any position in the domain but to a positive singular exponent.

Remark 7. For simplicity (dimensionless displacements, Remark 1), the definition of the intensity factor k_{III} (29) directly involves the displacement field (30) and (31). Thus, in the homogeneous case ($G_1 = G_3$) for instance, it differs from the usual definition by a coefficient $2/G_1$ (up to a constant $\sqrt{2\pi}$).

In-plane and out-of-plane components are uncoupled, thus, in the present case, U^0 defined in (2) and (3) undergoes the Mode III singular behaviour

$$U^0(x_1, x_2) = U^0(0, 0) + k^0_{\mathrm{III}}\sqrt{r}u_{\mathrm{III}}(\theta) + \hat{U}^0(x_1, x_2), \tag{32}$$

where $\hat{U}^0$ is a smooth part (i.e. involving exponents greater or equal to 1).

As a consequence, conditions (15), (20) and (28) are incompatible with the usual smoothness conditions which guarantee existence and uniqueness of the elastic solutions. This is the main feature of this paper, in case of a singular point, the second outer term U^1 cannot be directly determined by the previously defined matching process. A superimposition principle must be invoked.

Using (32), U^1 splits into two parts

$$U^1(x_1, x_2) = k^0_{\mathrm{III}}\bar{U}^1(x_1, x_2) + \hat{\bar{U}}^1(x_1, x_2). \tag{33}$$

The second one is the solution to a well-posed problem with boundary conditions derived from the regular part $\hat{U}^0$ of U^0, whereas $\bar{U}^1$ and $\bar{\sigma}^1$ are associated with the boundary and transmission conditions derived from the singular part of (32)

$$[\![\bar{U}^1]\!] = \frac{1}{\sqrt{r}}\frac{2G_1G_3 - G_2(G_1 + G_3)}{4G_2G_3}, \quad [\![\bar{\sigma}^1]\!]\underline{n} = 0 \quad \text{through } \Gamma^{13}_{\mathrm{u}} \quad (r = -x_1),$$

$$G_1\frac{\partial \bar{U}^1_+}{\partial x_2} = \frac{1}{4r\sqrt{r}}H_+(d) \quad \text{along } \Gamma^{13}_{\mathrm{c}+} \quad (r = x_1),$$

$$G_3\frac{\partial \bar{U}^1_-}{\partial x_2} = -\frac{G_1}{G_3}\frac{1}{4r\sqrt{r}}H_-(d) \quad \text{along } \Gamma^{13}_{\mathrm{c}-} \quad (r = x_1).$$

This term is a priori consistent with a strongly singular behaviour $1/\sqrt{r}$ and a splitting

$$\bar{U}^1(x_1, x_2) = \frac{1}{\sqrt{r}}v(\theta) + \bar{\bar{U}}^1(x_1, x_2), \tag{34}$$

where $\bar{\bar{U}}^1$ is a complementary part, smoother than the first term. Unfortunately, it is impossible to determine the function $v(\theta)$ such that the above first term fits into the singular part of the boundary conditions. This relies on the Fredholm theorem (Leguillon and Sanchez-Palencia, 1987, p. 179). The exponent 1/2 is characteristic of the problem and as a 2-D property −1/2 is too, thus determining v associated with this last exponent needs to solve a non-homogeneous problem with a non-invertible operator. The operator kernel contains the dual mode $U_{\mathrm{III}-}$ to the already mentioned mode $U_{\mathrm{III}+}$ (30) (Leguillon and Sanchez-Palencia, 1987, p. 91)

$$U_{\mathrm{III}-}(x_1, x_2) = \frac{1}{\sqrt{r}} u_{\mathrm{III}}(\theta). \tag{35}$$

As a very specific property of this situation (scalar problem) the same function $u_{\mathrm{III}}(\theta)$ (31) is involved in the two dual modes but this is not a general rule.

The problem can admit an infinite number of solutions provided some compatibility conditions, but there are not satisfied herein. Thus (34) is wrong, the solution must be sought in the form (Leguillon and Sanchez-Palencia, 1987, p. 182)

$$\bar{U}^1(x_1, x_2) = \frac{1}{\sqrt{r}}[C \ln r\, u_{\mathrm{III}}(\theta) + v(\theta)] + \bar{\bar{U}}^1(x_1, x_2). \tag{36}$$

The constant C is selected such that the previously mentioned compatibility conditions hold true. Moreover, if calculations lead to $C = 0$, (36) reduces to (34), it proves that the initial compatibility condition was fulfilled.

Let us focus our attention on the first term of the right-hand-side member of (36). From the equilibrium equation (in polar coordinates) and the jumps conditions through Γ^{13} (15) and (20), Eq. (36) becomes

$$\frac{1}{\sqrt{r}}(C \ln r\, G'_{\pm} \cos(\theta/2) + v_{\pm}(\theta)) = \frac{G'_{\pm}}{\sqrt{r}}(C \ln r \cos(\theta/2) + C\theta \sin(\theta/2) + A_{\pm} \sin(\theta/2) + B \cos(\theta/2)), \tag{37}$$

with

$$G_3 A_+ - G_1 A_- + (G_3 - G_1) C\pi = \frac{G_1 - G_2}{4G_2} G_3 + \frac{G_3 - G_2}{4G_2} G_1.$$

Here, $G'_+ = 1$ and $G'_- = G_1/G_3$, and as usual the index $\pm$ holds for the upper ($x_2 > 0$) and lower ($x_2 < 0$) half planes, respectively.

The boundary conditions on the crack lips lead to

$$A_+ = \frac{H_+(d)}{2G_1}, \quad A_- = \frac{H_-(d)}{2G_3} - 2C\pi.$$

At this time, the constant B remains unknown, it means exactly that the solution is determined up to one element (35) of the operator kernel.

5. Matched asymptotics around the crack tip

The above relations (32), (33) and (36) express the behaviour of the solution when approaching the singular crack tip, where the previous expansions are questionable. To be consistent, after the change of variable $y = x/\varepsilon$, there would exist an inner expansion whose behaviour at infinity is governed by the above mentioned terms

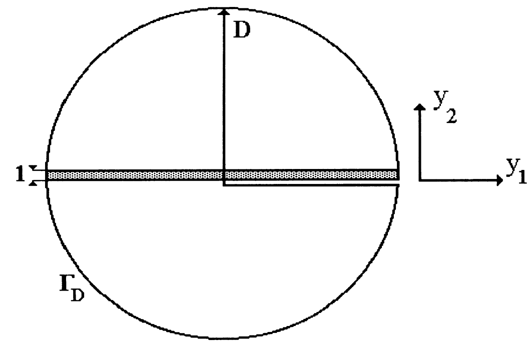

Fig. 8. The inner domain derived from stretching $y_1 = x_1/\varepsilon$, $y_2 = x_2/\varepsilon$.

$$U^{\varepsilon}(\varepsilon y_1, \varepsilon y_2) = \Phi_0(\varepsilon) W^0(y_1, y_2) + \Phi_1(\varepsilon) W^1(y_1, y_2) + \Phi_2(\varepsilon) W^2(y_1, y_2) + \cdots.$$

As usual, the first term matching is easy

$$\Phi_0(\varepsilon) = 1, \quad W^0(y_1, y_2) = U^0(0, 0).$$

The next term leads to a decaying condition at infinity

$$\Phi_1(\varepsilon) = k^0_{\mathrm{III}} C\sqrt{\epsilon}\ \ln\ \varepsilon, \quad W^1(y_1, y_2) \sim \frac{1}{\sqrt{\rho}} u_{\mathrm{III}}(\theta) \quad \text{as } \rho \to \infty \quad (\rho = r/\varepsilon). \tag{38}$$

The third matching condition has contributions from various terms of the outer expansion

$$\Phi_2(\varepsilon) = k^0_{\mathrm{III}}\sqrt{\varepsilon}, \quad W^2(y_1, y_2) \sim \sqrt{\rho} u_{\mathrm{III}}(\theta) + \frac{1}{\sqrt{\rho}}(C \ln \rho\ u_{\mathrm{III}}(\theta) + v(\theta) + \cdots). \tag{39}$$

Obviously, in (39) the behaviour at infinity is governed by the first term, i.e. the crack tip Mode III, the corresponding solution can be sought by superimposition in the form

$$W^2(y_1, y_2) = \sqrt{\rho} u_{\mathrm{III}}(\theta) + \hat{W}^2(y_1, y_2), \tag{40}$$

and $\hat{W}^2$ is the solution to a well-posed problem, its behaviour at infinity is exactly described by the terms of (39) missing in (40).

Remark 8. This point is not really obvious. An option is to consider the numerical approach used to solve these inner problems. The infinite domain is artificially bounded at a large distance $D \gg 1$ (i.e. large compared to the stretched layer thickness) (Fig. 8), this new created boundary Γ_D supporting the homogeneous Dirichlet condition

$$W^2_D(y_1 y_2) = \sqrt{\rho} u_{\mathrm{III}}(\theta) \quad \text{on } \Gamma_D.$$

This allows one to avoid any difficulty related to the conditions at infinity and as $D \to \infty$, the computed solution W^2_D converges (up to an irrelevant constant) toward the required solution W^2 (Sanchez-Palencia, 1995).

But W^1 cannot be solved in the same way, no superimposition process can be invoked and the problem has in general, no solution. The unknown constant B in (37) can be used to overcome this difficulty. Taking

$$B = -C \ln \varepsilon$$

allows the cancellation of the hindering term W^1 (38). Then, the inner expansion reads

$$U^{\varepsilon}(\varepsilon y_1, \varepsilon y_2) = U^0(0, 0) + \sqrt{\varepsilon} k^0_{\mathrm{III}}(\sqrt{\rho} u_{\mathrm{III}}(\theta) + \hat{W}^2(y_1, y_2)) + \cdots, \tag{41}$$

where $\hat{W}^2$ is defined by (40).

A consistent outer expansion must be rewritten

$$U^{\varepsilon}(x_1, x_2) = U^0(x_1, x_2) - \varepsilon \ln \varepsilon C k^0_{\mathrm{III}} U^1(x_1, x_2) + \varepsilon k^0_{\mathrm{III}} U^2(x_1, x_2) + \cdots. \tag{42}$$

The first term of the above expansion is described by (32) and the next one by

$$U^1(x_1, x_2) = \frac{1}{\sqrt{r}} u_{\mathrm{III}}(\theta) + \hat{U}^1(x_1, x_2).$$

Without omitting the smooth term from (33), the third one reads

$$U^2(x_1, x_2) = \frac{1}{\sqrt{r}}(C \ln r u_{\mathrm{III}}(\theta) + \hat{v}(\theta)) + \hat{U}^2(x_1 x_2) + \hat{\hat{U}}^1(x_1, x_2),$$

where $\hat{v}(\theta) = v(\theta) - B u_{\mathrm{III}}(\theta)$, or in other words, using (37)

$$\hat{v}(\theta) = G'_{\pm}(C\theta + A_{\pm}) \sin(\theta/2).$$

The complementary terms $\hat{U}^1$, $\hat{U}^2$ and $\hat{\hat{U}}^1$ are solutions to well-posed problems defined below in the numerical example.

Remark 9. This reasoning is not a proof of the existence of such expansions, there is simply no counter argument and the results seem to be in logical agreement with the simpler case of a singular point like a corner (Leguillon, 1994, 1995) and with more conventional situations (Leguillon, 1993).

There are some particular cases to exhibit.

- First, if the two substrates are identical, $G_1 = G_3$, it is the situation examined by Atkinson and Chen (1996). The coefficient C is independent of d

$$C = \frac{G_1^2 - G_2^2}{4\pi G_1 G_2}. \tag{43}$$

  It is positive if the adhesive layer is more compliant than the substrates and negative otherwise. It vanishes for $G_2 = G_1$, when there is no bonding zone (the three domains are made of the same material, $G_1 = G_2 = G_3$). The outer expansion change due to the crack location occurs only in the third term through the coefficients $A_{\pm}$.
- Second, in the case considered by Hutchinson et al. (1987), if there is no adhesive layer. The two materials $\Omega^{1\varepsilon}$ and $\Omega^{3\varepsilon}$ are in contact along the interface $\Gamma^{13\varepsilon}$ located at $x_2 = \varepsilon/2$ for instance, then $G_2 = G_3$. The crack is paralleling the interface and

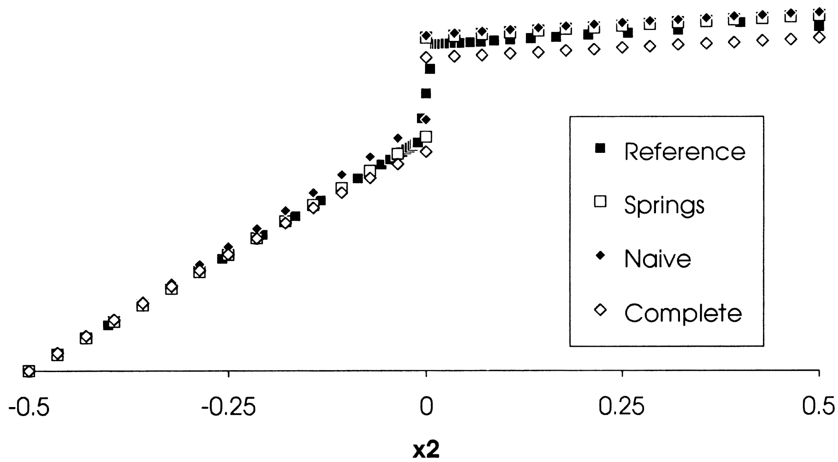

Fig. 9. Out-of-plane displacement on the left side of the square, comparison between an FE reference solution and three different models.

$$C=\frac{G_1-G_3}{4\pi G_1}(1-2d)\quad \text{if } d\leq 1/2 \quad\text{and}\quad C=\frac{G_1-G_3}{4\pi G_3}(1-2d)\quad \text{if } d\geq 1/2,$$

the crack is below the interface in the first case and above in the second one. The coefficient C vanishes for $d = 1/2$, i.e. for an interface crack, and for $G_1=G_3$ when as before there is no interface at all.

6. Numerical example

Numerical tests have been performed to illustrate the above results. Let us consider a 1×1 dimensionless square Ω (Fig. 8) made of two materials Ω^1 and Ω^3 ($G_1=10$ GPa, $G_3=1$ GPa) bonded together along an interface Γ^{13} located at $x_2=0$. A crack (length$=0.3$) is lying along the interface, starting from the right edge. The structure is clamped on the bottom face and a unit out-of-plane traction $T = 1$ GPa acts on the upper face (thus, for a given traction T' it suffices to multiply displacements, stresses and load dependent intensity factors by T').

With the help of a fine mesh in the vicinity of the joint, a first computation taking into account a dimensionless joint thickness $\varepsilon=0.02$ and a compliant adhesive material $G_2=0.1$ GPa is used as reference (Fig. 9).

Next the springs model is checked. It does not require a fine mesh, since there is no joint in the unperturbed structure, but necessitates to introduce sliding elements along the interface, i.e. the axis $x_2=0$ (Léné and Leguillon, 1982), with a spring stiffness $G=G_2/\varepsilon=5$ GN/m.

Finally, the present model is put to work from two points of view. The simplest 'naive' one consists of ignoring the theoretical difficulty resulting from the singular behaviour of U^0 at the crack tip (Section 4). Indeed, the finite elements approximation yields (mesh dependent) bounded derivatives which can be directly introduced in the transmission condition (15), moreover, using linear elements (three nodes Lagrange triangles) cancels the second transmission condition (20). The second more rigorous technique implies the calculation of each term of the outer expansion (42). The leading term U^0 is the solution to the classical problem with perfect transmission conditions through the interface. Then, the complementary parts of the corrective terms are determined. The boundary conditions to apply are

$$\hat{U}^1(x_1, x_2) = -\frac{1}{\sqrt{r}} u_{\mathrm{III}}(\theta), \tag{44}$$

$$\hat{U}^2(x_1, x_2) = -\frac{1}{\sqrt{r}} (C \ln r\, u_{\mathrm{III}}(\theta) + \hat{v}(\theta)),$$

$$\hat{\bar{U}}^1(x_1, x_2) = 0, \tag{45}$$

on the bottom edge, where displacements are prescribed in the original problem.

$$\sigma(\hat{U}^1)\underline{n} = 0, \quad \sigma(\hat{U}^2)\underline{n} = 0, \quad \sigma(\hat{\bar{U}}^1)\underline{n} = 0,$$

on the crack lips [$\sigma(\varphi)$ denotes the stress field associated with the displacement φ].

$$\sigma(\hat{U}^1)\underline{n} = -\sigma\left(\frac{1}{\sqrt{r}} u_{\mathrm{III}}(\theta)\right)\underline{n} \tag{46}$$

$$\sigma(\hat{U}^2)\underline{n} = -\sigma\left[\frac{1}{\sqrt{r}} (C \ln r\, u_{\mathrm{III}}(\theta) + \hat{v}(\theta))\right]\underline{n}$$

$$\sigma(\hat{\bar{U}}^1)\underline{n} = 0 \tag{47}$$

on the upper, right and left edges of the structure, where forces are prescribed in the original problem. The above conditions (44)–(47) involving the strongly singular term $1/\sqrt{r}$ are prescribed only at a distance of the origin, thus, as expected, it avoids any difficulty. These conditions are completed for the first two terms by usual transmission conditions through the interface

$$[\![\hat{U}^1]\!] = 0, \quad [\![\hat{U}^2]\!] = 0, \quad [\![\sigma(\hat{U}^1)]\!]\underline{n} = 0, \quad [\![\sigma(\hat{U}^2)]\!]\underline{n} = 0.$$

Concerning the transmission conditions valid for $\hat{\bar{U}}^1$, the previous reasoning (applied to the so-called 'naive' method) is used leading to

$$[\![\hat{\bar{U}}^1]\!] = \frac{G_1 - G_2}{2G_2} \frac{\partial(U_+^0 - k_{\mathrm{III}}^0 U_{\mathrm{III}+})}{\partial x_2}(x_1, 0) + \frac{G_3 - G_2}{2G_2} \frac{\partial(U_-^0 - k_{\mathrm{III}}^0 U_{\mathrm{III}+})}{\partial x_2}(x_1, 0), \quad [\![\sigma(\hat{\bar{U}}^1)]\!]\underline{n} = 0.$$

This complete model seems to be a little bit stiffer than the reference while the other two (naive and springs) are slightly more compliant. We point out in this section that, computations have been performed on the unperturbed structure, as a consequence the exact location of the crack does not play any role. It can be within the joint, along the upper or lower faces or even outside at a short distance, it is modelled by the same interface crack. The only changes in the outer expansion occur through the coefficient C and the function $\hat{v}$.

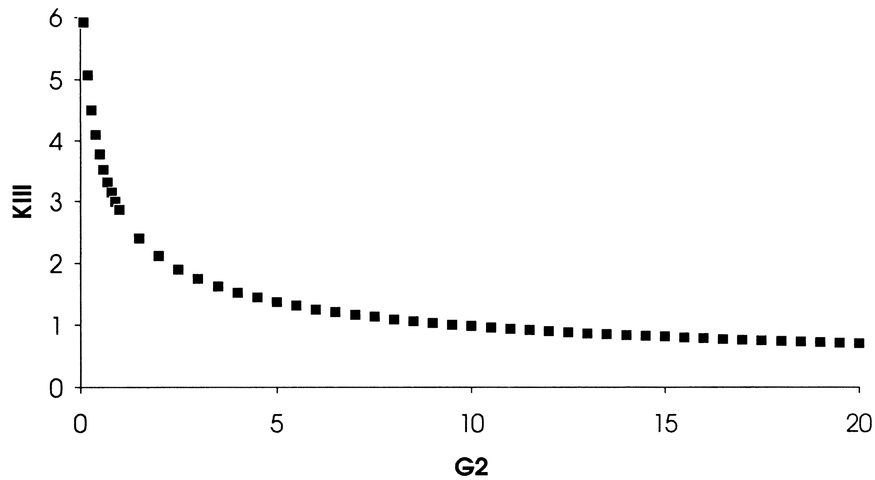

Fig. 10. K_{III} vs G_2 for $G_1 = G_3 = 10$ GPa.

7. Fictitious and actual intensity factors

The unperturbed structure, undergoing problems defining the various terms of the outer expansion, is in a way a virtual one. The adhesive layer has disappeared and is replaced by an ideal line. It is possible to define a Mode III intensity factor k^0_{III}, k^1_{III}, ..., for each term U^0, $\hat{U}^1$, ..., of this outer expansion, they correspond to an interface crack singularity characterized by (30) and (31). As a consequence (42) provides a fictitious expansion

$$k^{\varepsilon}_{\text{III}} = k^0_{\text{III}}(1 - \varepsilon \ln \varepsilon \, Ck^1_{\text{III}} + \cdots). \tag{48}$$

Relations (43) and (48) are contradictory with Atkinson and Chen (1996) conclusions for instance. For $\varepsilon = 0.02$, $G_1 = G_3 = 10$ GPa, $G_2 = 0.1$ GPa, then $C \simeq 7.96$. Calculations by contour integrals (Leguillon and Sanchez-Palencia, 1987, p. 77) give

$$k^0_{\text{III}} = 0.079, \quad k^1_{\text{III}} = 1.31 \quad \text{and then } k^{\varepsilon}_{\text{III}} \simeq 1.62, \quad k^0_{\text{III}} = 0.128,$$

neglecting the other terms of the expansion. Thus, from (48), one would derive first that the intensity factor increases with joint thickness and second, that an adhesive layer more compliant than the substrates tends to increase the intensity factor of the crack tip singularity. This last point is not satisfactory, Atkinson and Chen (1996) state the contrary (which seems more rational), but as seen below, the terms involved in (48) are meaningless.

The leading term k^0_{III} of this expansion is baptised the fictitious intensity factor, it is not associated with any existing singularity of the exact structure but only with a singularity of the simplified frame. It is, in a way, meaningless although it is typically the intensity factor computed by a classical finite element approach in structural computations. It is different from the actual one derived from (41)

$$k^{\varepsilon}_{\text{III}} = k^0_{\text{III}} K_{\text{III}} + \cdots,$$

where K_{III} is the Mode III intensity factor associated with W^2 (40) and expressed in terms of the stretched variable $\sqrt{\rho} = \sqrt{r/\epsilon}$. As well as k^1_{III} it is independent of the external applied loads. Moreover, involved in an inner problem it is also independent of the exact and eventually complicated geometry of the full structure (see Remark 8 and Fig. 8). Thus, at the leading-order, there exists a linear relation between the two coefficients k_{III} and k^{0}_{III} as also shown by Hutchinson et al. (1987) in a slightly different situation

$$k_{\mathrm{III}} = k^{0}_{\mathrm{III}} K_{\mathrm{III}}. \tag{49}$$

The corresponding singularity to take into account in this case really exists and depends on the location of the crack tip in the stretched inner domain, it is characterized by

$$V_{\mathrm{III}}(y_1^*, y_2^*) = \sqrt{\rho^*}\, v_{\mathrm{III}}(\theta^*),$$

where the superscript * denotes Cartesian and polar coordinates with the origin at the crack tip. Fig. 10 plots the multiplicative coefficient K_{III} in the case $G_1 = G_3 = 10$ GPa for different values of G_2 from 0.1 to 20 GPa (i.e. from $G_1/100$ to $2G_1$) and for a crack lying in the middle of the joint. Coefficient K_{III} equals 1 when $G_1 = G_2 = G_3$ as expected. It is computed from a finite element solution W_D^2 (Remark 8) by contour integrals (Leguillon and Sanchez-Palencia, 1987), as already mentioned above.

Remark 10. In this specific homogeneous case with the crack lying within the joint, K_{III} can be analytically derived by equating the remote and local energy release rates (Fleck et al., 1991, Atkinson and Chen, 1996)

$$\frac{G_1}{2}\left(k^{0}_{\mathrm{III}}\right)^2 = \frac{G_2}{2} k^2_{\mathrm{III}}, \tag{50}$$

leading to

$$K_{\mathrm{III}} = \sqrt{\frac{G_1}{G_2}},$$

which is in complete agreement with Fig. 10. The unusual relation (50) is due to the slightly modified intensity factors definition (Remark 7), k^{0}_{III} *and* k_{III} differ from the usual ones, respectively, by $2/G_1$ and $2/G_2$.

The situation becomes more and more entangled depending on the location of the crack. If the crack tip lies strictly in one of the domains Ω^k, $k = 1, 2, 3$, the same function is involved, it corresponds (as in the above example) to a crack in a homogeneous material

$$v_{\mathrm{III}}(\theta^*) = \cos(\theta^*/2).$$

If the crack lies along the interface between the upper substrate and the adhesive layer, the definition is different, it depends on the properties of the adjacent materials

$$v_{\mathrm{III}}(\theta^*) = \cos(\theta^*/2) \quad \text{in } \Omega^1, \qquad v_{\mathrm{III}}(\theta^*) = \frac{G_1}{G_2}\cos(\theta^*/2) \quad \text{in } \Omega^2.$$

Finally, if the crack lies at the bottom, along the interface between the adhesive layer and the lower substrate, there is a similar definition, but depending on the properties of other adjacent materials

$$v_{\mathrm{III}}(\theta^*) = \cos(\theta^*/2) \quad \text{in } \Omega^2, \qquad v_{\mathrm{III}}(\theta^*) = \frac{G_2}{G_3}\cos(\theta^*/2) \quad \text{in } \Omega^3.$$

The multiplicative coefficient K_{III} can be defined and computed (as above) whatever the relative stiffness of the joint, it can be smaller (as often) or larger than the substrates stiffness. It is not possible to make the same analysis with a springs model which works only in case of very low joint stiffness.

8. Conclusion

The mismatch between the fictitious and actual intensity factors is obvious, it is not the same singular modes which are involved. Moreover, it is vain to expect a better coherency from the energy release rate G. This one is defined as a limit

$$G = \lim_{\delta\ell \to 0} -\frac{\delta W}{\delta\ell},$$

where δW is the change in potential energy corresponding to a crack increment δl. This definition implies to use crack increments as small as needed and thus, becoming smaller than the joint thickness itself, but such a calculation is obviously not permitted for the outer terms which are insensitive to very small δl (smaller than e). Any energy release rate computed from the outer terms and especially from the leading one U^0 is then meaningless. A similar conclusion is drawn by Ryvkin et al. (1995). Their far intensity factor corresponds to the present k^0_{III} and their near one to k_{III} [see (49)].

Atkinson and Chens' (1996) case is special, the crack lies in a homogeneous material in both local and remote models. Then, the same displacement mode (30) and (31) is involved and there is simply a stress rescaling by G_1 or G_2 leading to the conclusion of Remark 10.

Remark 11. All these considerations lead to define an apparent toughness of the interface to be used in a finite element simulation (i.e. based on the computation of U^0)

$$k^{\mathrm{app}}_{\mathrm{IIIc}} = \frac{k_{\mathrm{IIIc}}}{K_{\mathrm{III}}}, \tag{51}$$

where k_{IIIc} is the toughness of the material forming the joint if the crack lies within the joint, of one of the substrates if the crack lies out of the joint, or one of the interfaces if the crack is at the top or the bottom of the joint. Not a single definition exists of the apparent interface toughness. With (51), the growth criterion, expressed in terms of the fictitious intensity factor, writes

$$k^0_{\mathrm{III}} \geq k^{\mathrm{app}}_{\mathrm{IIIc}}.$$

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